- Production $F(K,L)$ with constant returns to scale,

$$Y = F(K,L), \quad \frac{Y}{L} = y = \frac{F(K,L)}{L} = F\left(\frac{K}{L}, 1\right) := f(k),$$

where $k = \frac{K}{L}$.

$$c = \frac{C}{L}$$

- Equation of motion - for capital

$$\dot{K} = Y - C - \delta K \quad \text{where } \delta \geq 0$$

- Assuming constant population growth, $\frac{\dot{L}}{L} = n \Rightarrow \underline{L(t) = L_0 e^{nt}}$
- Equation of motion for $k = \frac{K}{L}$

$$\dot{k} = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} = \frac{\dot{K}}{L} - nk$$

$$\underline{\dot{k} = y - c - (\delta + n)k}$$

Optimal economic growth model (Ramsey-Cass-Koopmans)

$$\max_{\{c(t)\}} \int_0^\infty e^{-\rho t} \cdot L(t) u(c(t)) dt = L_0 \int_0^\infty e^{-(\rho - n)t} u(c(t)) dt$$

s.t. $\dot{k} = y - c - (\delta + n)k$, with k_0 - given.

The current value Hamiltonian

$$\mathcal{H}^c = u(c) + \lambda\left(y - c - (\delta + n)k\right)$$

FOCs:

$$\begin{cases} \frac{\partial \mathcal{H}^c}{\partial c} = u'(c) - \lambda = 0 \\ \frac{\partial \mathcal{H}^c}{\partial k} = \lambda\left(f'(k) - (\delta+n)\right) = (\rho - n)\cdot\lambda - \dot{\lambda} \\ \dot{k} = y - c - (\delta+n)k \end{cases}$$

$$\dot{\lambda} = -\lambda\left(f'(k) - \delta - \rho\right) \Leftrightarrow \boxed{\frac{\dot{\lambda}}{\lambda} = -\left(f'(k) - \delta - \rho\right)}$$

$$\dot{\lambda} = u''(c)\cdot\dot{c} \Leftrightarrow \frac{\dot{\lambda}}{\lambda} = \frac{u''(c)\,\dot{c}}{u'(c)}$$

The Euler equation

$$-\frac{u''(c)}{u'(c)}\cdot\dot{c} = f'(k) - \delta - \rho$$

Digression

Def. $RRA = \theta = -\frac{u''(c)}{u'(c)}c$ – relative risk aversion

Def. Function of CRRA class – constant RRA

$$\begin{cases} u(c) = \frac{c^{1-\theta} - 1}{1-\theta} & \text{if } \theta > 0,\ \theta \neq 1 \\ u(c) = \ln c & \text{if } \theta = 1 \end{cases}$$

Here, if we assume CRRA utility, the Euler equation becomes

$$\boxed{\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{\theta}}$$

Dynamic system of the Ramsey model

$$\begin{cases} \dot{c} = c \cdot \frac{1}{\theta} \cdot (f'(k) - \delta - \rho) \\ \dot{k} = f(k) - c - (\delta + n)k \end{cases}$$

Steady state $\begin{cases} \dot{k} = 0 \\ \dot{c} = 0 \end{cases} \iff \begin{cases} f'(k) = \delta + \rho \\ c = f(k) - (\delta + n)k \end{cases} \Rightarrow \boxed{\text{a unique } (c^*, k^*)}$

Phase diagram

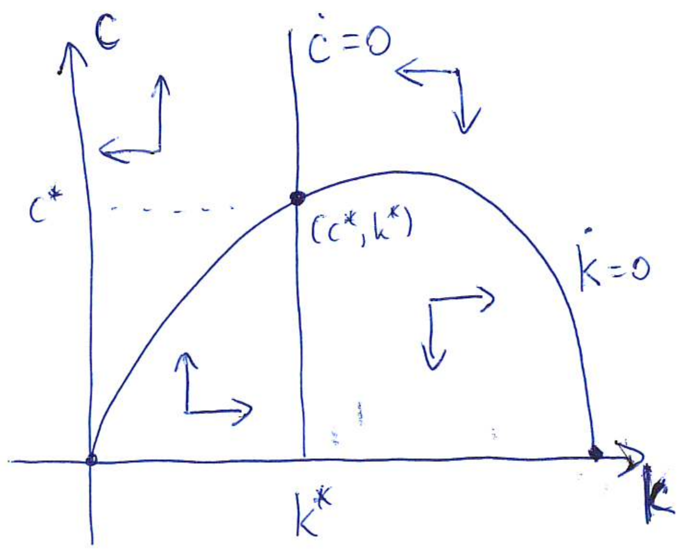
Isoclines

1° $\dot{c} = 0 \iff f'(k) - \delta - \rho = 0$

→ if $c > 0$

→ meaning that $k = k^*$

2° $\dot{k} = 0 \iff c = f(k) - (\delta + n)k$



TRANSVERSALITY CONDITIONS

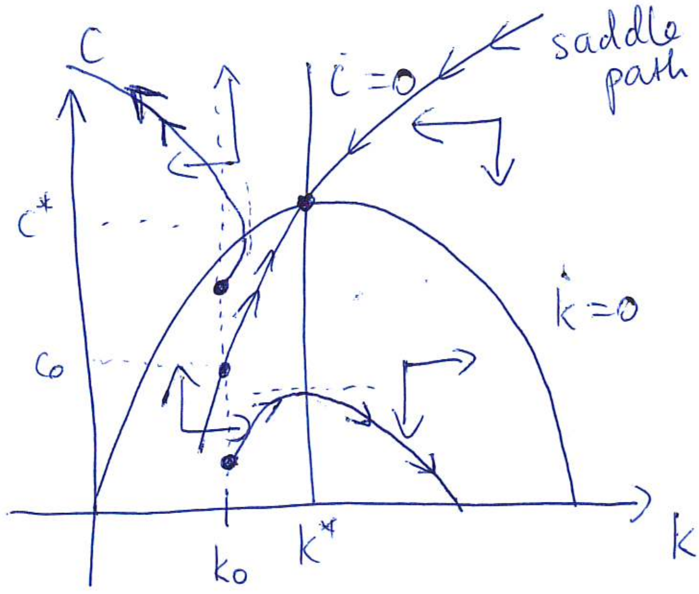
for infinite-horizon problems (with continuous time)

- $\lim\limits_{t\to\infty} e^{-\beta t}\lambda(t) = 0$
- $\lim\limits_{t\to\infty} \left| e^{-\beta t}\lambda(t)\, s(t)\right| < \infty \iff \int_0^\infty e^{-\beta t} u(c(t))\,dt < \infty$ (integrability)

(if λ - associated with the current-value Hamiltonian)

(back to the Ramsey model)

- $\lim\limits_{t\to\infty} e^{-\beta t}\lambda(t) = \lim\limits_{t\to\infty} u'(c)\cdot e^{-\beta t} = 0$ with $\beta = \rho - n > 0$
- $\lim\limits_{t\to\infty} \left| e^{-\beta t}\lambda(t)k(t)\right| = \lim\limits_{t\to\infty} e^{-(\rho-n)t} u'(c)\cdot k$ - should be finite.



Only the saddle path is consistent with the TVC.
In the optimum, one has to choose c_0 to follow the saddle path.

Human capital accumulation model

$$\max_{\{c(t)\}} \int_0^\infty e^{-\rho t} u(c)\, dt \quad \text{s.t. } y = hl, \quad c = y,$$
$$\dot{h} = A(1-l)h^{\gamma} - \delta h$$

with $\rho > 0$, $\gamma \in (0,1)$, $\delta > 0$, $A > 0$, and $h \geq 0$, $l \in [0,1]$.

Let us assume CRRA utility, $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$. h_0 - given.

$$\mathcal{H} = e^{-\rho t} u(c) + \lambda \left(A(1-l)h^{\gamma} - \delta h \right) =$$

NOT THE CURRENT-VALUE $\mathcal{H}^c$!

$$= e^{-\rho t} \frac{(hl)^{1-\theta} - 1}{1-\theta} + \lambda \left(A(1-l)h^{\gamma} - \delta h \right)$$

FOCs

$$\frac{\partial \mathcal{H}}{\partial l} = e^{-\rho t} h^{1-\theta} l^{-\theta} - \lambda A h^{\gamma} = 0 \Rightarrow \boxed{\lambda = \frac{e^{-\rho t} h^{1-\theta-\gamma} l^{-\theta}}{A}}$$

$$\frac{\partial \mathcal{H}}{\partial h} = e^{-\rho t} h^{-\theta} l^{1-\theta} + \lambda \left(A(1-l)\gamma h^{\gamma-1} - \delta \right) = -\dot{\lambda}$$

Hence, $\hat{\lambda} = -\rho + (1-\theta-\gamma)\hat{h} - \theta \hat{l}$

$$-\hat{\lambda} = A(1-l)\gamma h^{\gamma-1} - \delta + \frac{e^{-\rho t} h^{-\theta} l^{1-\theta}}{\lambda} = l h^{\gamma-1} A$$

$$A(1-l)\gamma h^{\gamma-1} - \delta + l h^{\gamma-1} A = \rho - (1-\theta-\gamma)\hat{h} + \theta \hat{l}$$

$$\theta \hat{l} = A(1-l)\gamma h^{\gamma-1} - \delta + A l h^{\gamma-1} - \rho + (1-\theta-\gamma)\left(A(1-l)h^{\gamma-1} - \delta \right)$$

$$\theta \hat{l} = h^{\gamma-1}\left[(1-\theta)A(1-l) + Al \right] - \delta(2-\theta-\gamma) - \rho$$

$$\underbrace{\hat{l} = \frac{1}{\theta}\left[h^{\gamma-1} A(1-\theta+\theta l) - \delta(2-\theta-\gamma) - \rho \right]}$$ ← Euler eq.

Phase diagram

$$\begin{cases} \dot{l} = \dfrac{l}{\theta}\left[h^{\gamma-1} A\,(1-\theta+\theta l) - \delta(2-\theta-\gamma) - \rho\right] \\ \dot{h} = A(1-l)h^{\gamma} - \delta h \end{cases}$$

Steady state: $\dot{h} = \dot{l} = 0$

$\boxed{\text{Isoclines}}$ $\dot{l} = 0 \iff h^{\gamma-1}(1-\theta+\theta l)A = \delta(2-\theta-\gamma)+\rho$

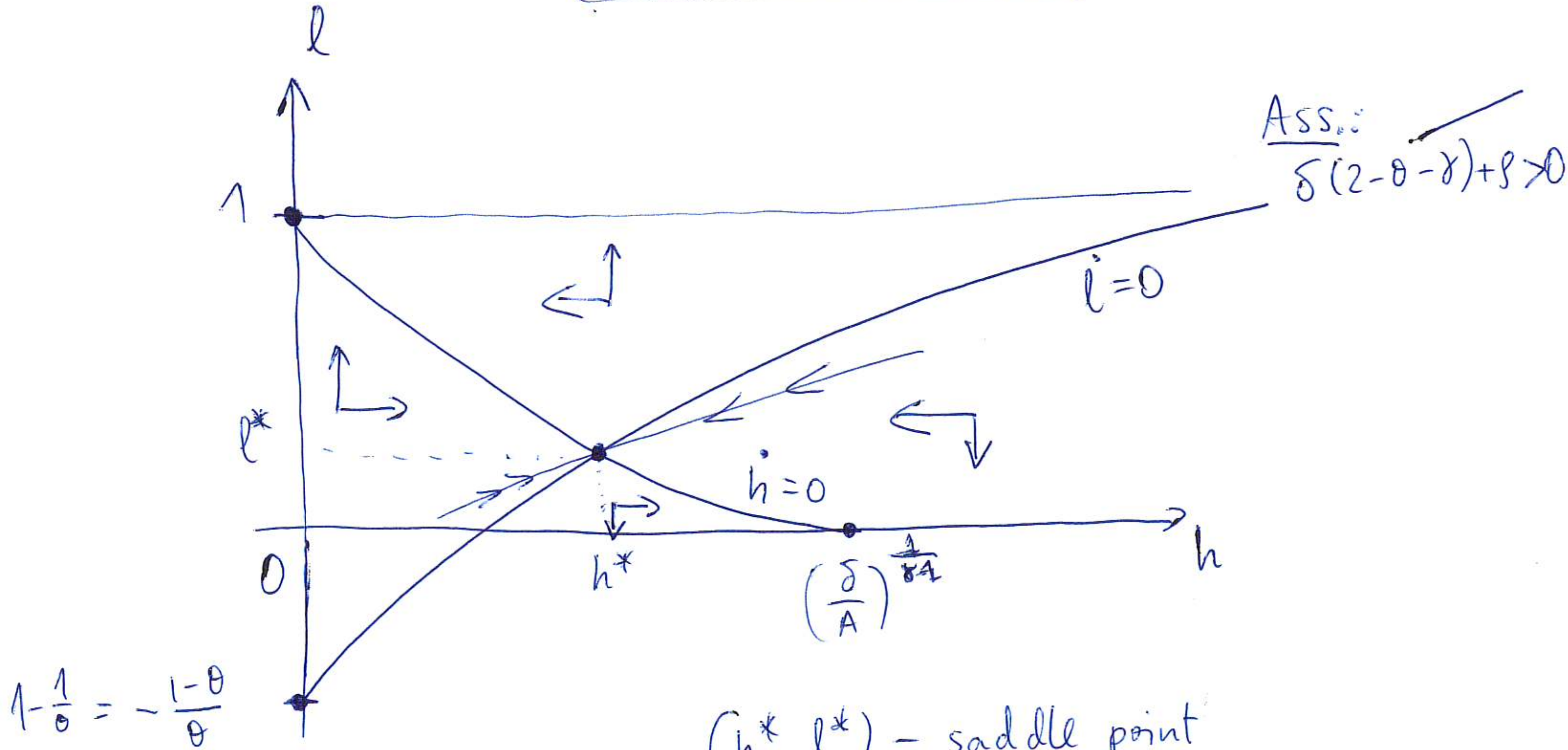
$$1-\theta+\theta l = \frac{h^{1-\gamma}}{A}\left(\delta(2-\theta-\gamma)+\rho\right)$$

$$l = \frac{1}{\theta}\left[\frac{\delta(2-\theta-\gamma)+\rho}{A}\cdot h^{1-\gamma} - (1-\theta)\right]$$

$\dot{h} = 0 \iff A(1-l)h^{\gamma-1} = \delta$

$$1-l = \frac{\delta}{A}h^{1-\gamma}$$

$$l = 1 - \frac{\delta}{A}h^{1-\gamma}$$



(h^*, l^*) – saddle point

$$\boxed{\begin{gathered}\text{Steady state}\\ l^* = \frac{1-\gamma+\rho/\delta}{2-\gamma+\rho/\delta}\\ h^* = \left(\frac{A}{\delta(2-\gamma)+\rho}\right)^{\frac{1}{1-\gamma}}\end{gathered}}$$

Transversality conditions

$$\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} \frac{e^{-\rho t}\, l^{-\theta}\, h^{1-\theta-\gamma}}{A} = 0$$

$$\lim_{t\to\infty} |\lambda(t) h(t)| = \lim_{t\to\infty} \frac{e^{-\rho t}\, l^{-\theta}\, h^{2-\theta-\gamma}}{A} \quad \text{— finite}$$

- satisfied if $l \to l^*$, $h \to h^*$ (saddle path)

Useful hint:

GROWTH RATE

$$\frac{\dot{(A^\alpha B^\beta)}}{A^\alpha B^\beta} = \widehat{A^\alpha B^\beta} = \alpha \hat{A} + \beta \hat{B}$$

$$\widehat{e^{-\rho t}} = \frac{(e^{-\rho t})'}{e^{-\rho t}} = \frac{-\rho e^{-\rho t}}{e^{-\rho t}} = -\rho$$